Rudin (7.16) FTC

$$f(x) - f(a) = \int_a^x f'(t)\,dt \qquad -(1)$$

- When does this hold? $f' \in L^1$: No, see Devil's staircase.

- ★ Note: f' need not be L^1 for (1) to hold
  $f(x) = x^2 \sin\left(\frac{1}{x^2}\right)$ $\quad \int_0^1 |f'| = +\infty$. But $f(x) - f(0) = \lim_{a \downarrow 0} \int_a^x f'(t)\,dt$.

  Very similar to midterm question.

Absolute continuity : $\forall \epsilon > 0 \ \exists \delta$ st $\forall n \in \mathbb{Z}^+$

$I = I_1 \sqcup \cdots I_n$ st $m(I) < \delta$

$$\sum |f(b_i) - f(a_i)| < \epsilon$$

TFAE: Let $f : I \to \mathbb{R}$ be continuous & nondecreasing.

what's a good example of a fn that doesn't.

1) f is AC

2) $m(E) = 0 \Rightarrow m(f(E)) = 0$ (Maps null sets to null sets)

3) f is diff a.e. on I $f' \in L^1$ and $f(x) - f(a) = \int_a^x f'(t)\,dt$

Pf: 3) ⇒ 1) (was on midterm)

1) ⇒ 2) Take any $m(E) = 0$ $\exists\, O$ (think outer measure) $E \subset O$ $m(O) < \delta$

Let $O = \bigsqcup_{i=1}^{\infty} I_i$ Then by AC $\sum_{i=1}^{n} f(b_i) - f(a_i) < \epsilon$ $\forall n$. Take $n \to \infty$

But $f(E) \subset f(O)$ and $f(O) = \bigcup_{i=1}^{\infty} f(I_i)$ From above $m(f(O)) \leq \epsilon$.

2) $\Rightarrow$ 3)

The idea is to define $\mu(E) = m(f(E))$. Then μ is countably additive if

$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(f(E_i))$ But for this $f(E_i) \cap f(E_j) = \emptyset$. But what if f is constant? Then this won't happen.

Let $g = x + f(x)$ Then g is strictly increasing and 1-1.

NEXT: Have to show $g(E)$ is meas. to find $m(g(E))$! Will use 2) for this!

g also satisfies 2) since for any I $m(g(I)) = m(f(I)) + |I|$.
g is continuous & strictly increasing. So by intermediate value theorem g maps I to a segment.
Then $g(b) - g(a) = f(b) - f(a) + b - a$

Then suppose $E \subset I$ is meas: $E = E_\sigma \cup E_0$ where $E_\sigma \in F_\sigma$
$\Rightarrow$ countable union of closed (and bounded) sets.

G_δ - countable inter of open sets
F_σ = " union of closed

If E is meas $m(E) = \sup\{m(F) : F \subset E, F \text{ closed}\}$.

$m(I \setminus E) = \inf\{m(O) : O \supset E^c\}$

$m(I) - m(E) \leq m(O) = m(I) - m(O^c) \Rightarrow m(E) \geq m(O^c)$.

So take a sequence of F_n st $m(F_n) \uparrow m(E)$ and $O_n \downarrow$ $m(O_n) \downarrow m(E)$

$m(E \setminus F_n) \leq m(O_n \setminus F_n) \longrightarrow 0$. Etc.

SKIP.

So $E = E_\sigma \sqcup E_0$ where $E_\sigma \in F_\sigma$ and E_0 is null.

$g(E_\sigma) = \bigcup_{i=1}^{\infty} g(F_i)$ and each $g(F_i)$ is compact as well.

If $x \in g(E_\sigma)$ Then $\exists y \in F_n$ $x = g(y)$. So $g(E_\sigma)$ is measurable. since it's a countable union of closed sets.

Then $g(E) = g(E_\sigma) \cup \underbrace{g(E_0)}_{\text{null}}$ is meas.

$\mu \ll m$ by 2) (g is AC)

By Radon-Nikodym we have $\exists\, h$ st

$$\mu(E) = \int_E h\,dx \qquad\qquad \mu((0,x)) = m(g(0,x)) = g(x) - g(0) = \int_0^x h\,dx$$

$$\Rightarrow \quad f(x) + x - f(0) - 0 = \int_0^x h\,dx \quad \Rightarrow \quad f(x) = \int_0^x (h-1)\,dx$$

By Lebesgue differentiation $f'(x) = h' - 1$ a.e m.

NOW to remove f is nondecreasing restriction.

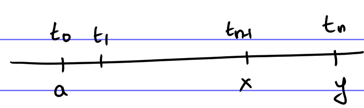
<u>Theorem</u>: Let $f:[a,b]\to\mathbb{R}$ Then let

$$F(x) = \sup_{P_{ab}} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \qquad\qquad P_{ab} \text{ partition}$$

be its total variation function. If f is AC, so are F, $F+f$ and $F-f$ and moreover they're nondecreasing.

If $F(b) < \infty$ we say f has bounded variation.

Pf: Let $x < y$ $\qquad F(y) \geq |f(y) - f(x)| + \sum_{i=1}^{n-1} |f(t_i) - f(t_{i-1})|$



$$\Rightarrow \quad F(y) \geq |f(y) - f(x)| + F(x) \qquad \text{taking sup over } P_{ax}$$

$$\Rightarrow \quad F(y) + f(y) \geq F(x) + f(x)$$

$$F(y) - f(y) \geq F(x) - f(x)$$

If $(\alpha,\beta) \subset I$ $\quad F(\beta) - F(\alpha) = \sup_{P_{\alpha\beta}} \sum |f(t_i) - f(t_{i-1})|$

Take any partn of $[a,\beta]$ and augment it to include α. Etc.

Now pick $\epsilon > 0$ $\exists\, \delta$ st $\forall \{I_j\}_{j=1}^n$ st $m(\bigcup_j I_j) < \delta$ $\sum |f(b_i) - f(a_i)| < \epsilon$

Then Take any partition of $\{I_j\}$ and

$$\sum_j \sum_{P_j} |f(t_{ij}) - f(t_{i-1,j})| \leq \epsilon$$

Now take a sup and get $\sum_j F(b_i) - F(a_i) \leq \epsilon$.

<u>Theorem</u>: If f is AC on I then f is differentiable on I, $f' \in L^1$ and

$$f(x) - f(a) = \int_a^x f'\, dt \qquad a.e\ x \in I = [a,b].$$

Pf: let $f_1 = \frac{1}{2}(F+f)$ $\quad f_2 = \frac{1}{2}(F-f)$. Apply previous theorem to f_1 & f_2 are AC and nondecreasing.

Then $f = f_1 - f_2$.

$$f(x) - f(a) = \int_0^x f_1'\, dt - \int_0^x f_2'\, dt$$

Remark: If $f(x) - f(a) = \int_a^x g$ for $g \in L^1$ Then f is AC. (Proved on midterm)

<u>Thm</u>: If f' exists EVERYWHERE and $f' \in L^1$ then $f(x) - f(a) = \int_a^x f'\, dt$

Not worth proving since requires 7.21